Estimating SAR polarimetric system errors in the presence of Faraday rotation using a transponder

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A PARC plays a key role in BIOMASS calibration

Selected site: New Norcia, NSW, Australia

Calibration roles:

- Characterisation of antenna pattern
- Radiometric calibration
- Polarimetric calibration

The system model

The basic system equation is:

$$\boldsymbol{M} = \begin{bmatrix} M_{hh} & M_{vh} \\ M_{hv} & M_{vv} \end{bmatrix}$$

$$= A(r,\theta)e^{j\varphi} \begin{bmatrix} 1 & \delta_2 \\ \delta_1 & f_1 \end{bmatrix} \begin{bmatrix} \cos\Omega & \sin\Omega \\ -\sin\Omega & \cos\Omega \end{bmatrix} \begin{bmatrix} S_{hh} & S_{vh} \\ S_{hv} & S_{vv} \end{bmatrix} \begin{bmatrix} \cos\Omega & \sin\Omega \\ -\sin\Omega & \cos\Omega \end{bmatrix} \begin{bmatrix} 1 & \delta_3 \\ \delta_4 & f_2 \end{bmatrix} + \begin{bmatrix} N_{hh} & N_{vh} \\ N_{hv} & N_{vv} \end{bmatrix}$$

More compactly, for polarimetric calibration:

$$\boldsymbol{M} = \boldsymbol{RFSFT} + \boldsymbol{N}$$

($\boldsymbol{N}$ here includes clutter)

With the PARC we can generate a known form of the $\boldsymbol{S}$ matrix.
Then, with no noise, on the right we have 13 real unknowns:

- 1 real unknown (Ω)
- 6 complex unknowns (the δ_i and ε_i, where $f_i = 1 + \varepsilon_i$)

On the left we have 4 complex (8 real) measurements.

Analytic solution without PARC errors

The PARC can simultaneously generate multiple targets with different scattering matrices

$$\mathbf{S}_X = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{S}_Y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{S}_A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{S}_B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Each independent target adds 8 new measurements so we have

- 13 real unknowns and 32 real measurements.

Hence many analytic ways to solve the noise-free system (e.g. Chen et al., 2011).

BUT, the PARC is not error-free, so when it tries to produce $\boldsymbol{S}$, it produces $\boldsymbol{S_D}$:

$$\begin{bmatrix} S_{hh} & S_{vh} \\ S_{hv} & S_{vv} \end{bmatrix}_D = \begin{bmatrix} 1 & \delta_{T2} \\ \delta_{T1} & f_{T1} \end{bmatrix} \begin{bmatrix} S_{hh} & S_{hv} \\ S_{vh} & S_{vv} \end{bmatrix} \begin{bmatrix} 1 & \delta_{T3} \\ \delta_{T4} & f_{T2} \end{bmatrix} \equiv \boldsymbol{T_P S R_P}$$

- 6 more complex unknowns, so 25 real unknowns in total.

With PARC errors we can solve the noise-free system using numerical methods

An analytic solution seems intractable but numerical optimisation can provide an exact solution for:

- the system errors
- the PARC errors
- Ω

1. What happens when we add clutter + noise?
2. Which performs better: the numerical approach or the analytic approach?

To assess this we used simulation:

System & PARC errors & noise were assumed to have:

- zero-mean complex Gaussian distributions
- independent real and imaginary parts with equal variance
- correlations only between δ_1 and δ_3, δ_2 and δ_4, and ε_1 and ε_2.

Slight detour: An improved empirical solution

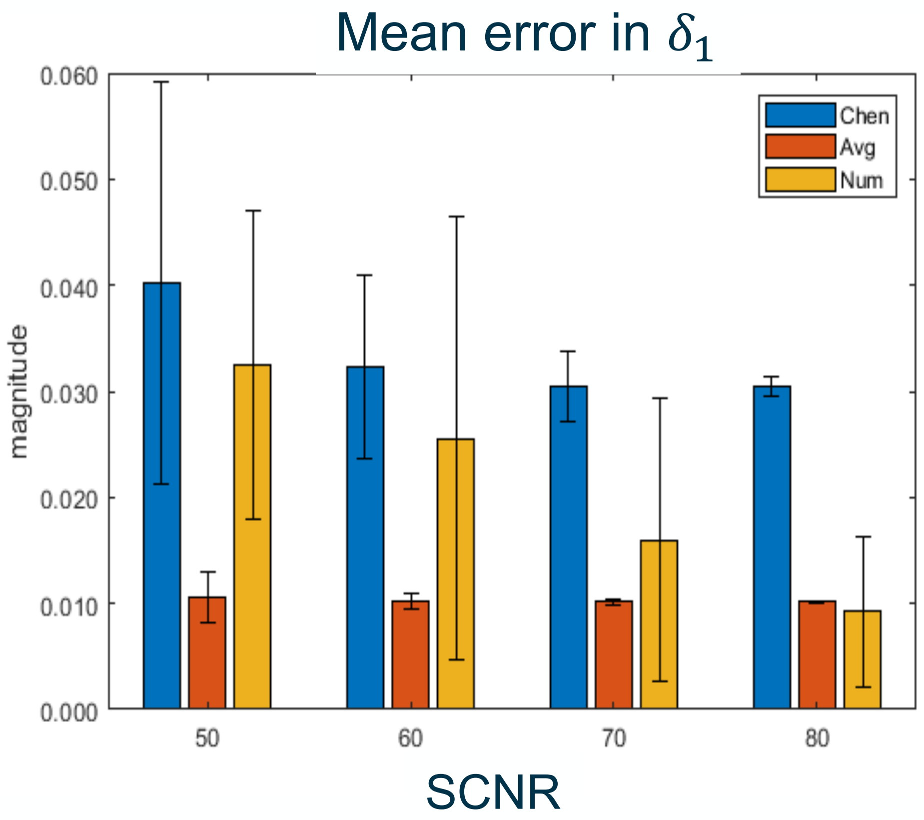
The first step in the Chen analytic solution is estimation of Ω.
This gives 2 solutions, and the one with the smallest imaginary part is selected.

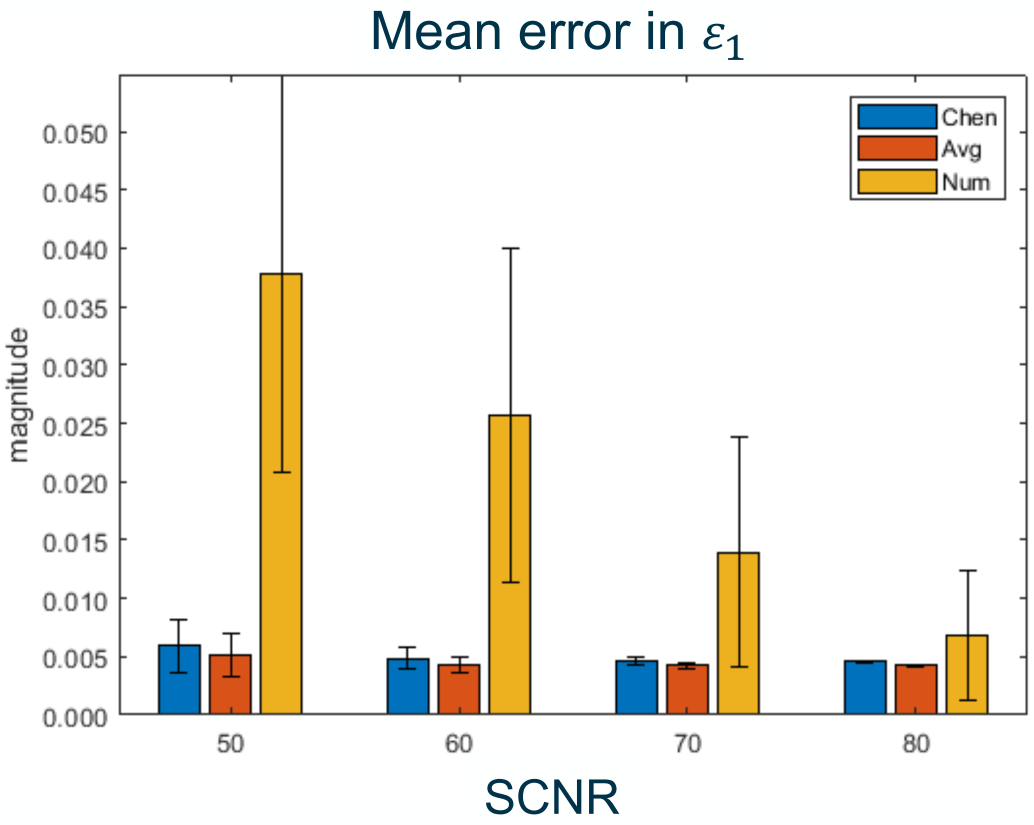
Mark noticed empirically that we got much better results by averaging the the 2 solutions for Ω and taking the real part.

The following results are therefore labelled as

- Chen
- Avg
- Num

Mean errors as SCNR changes

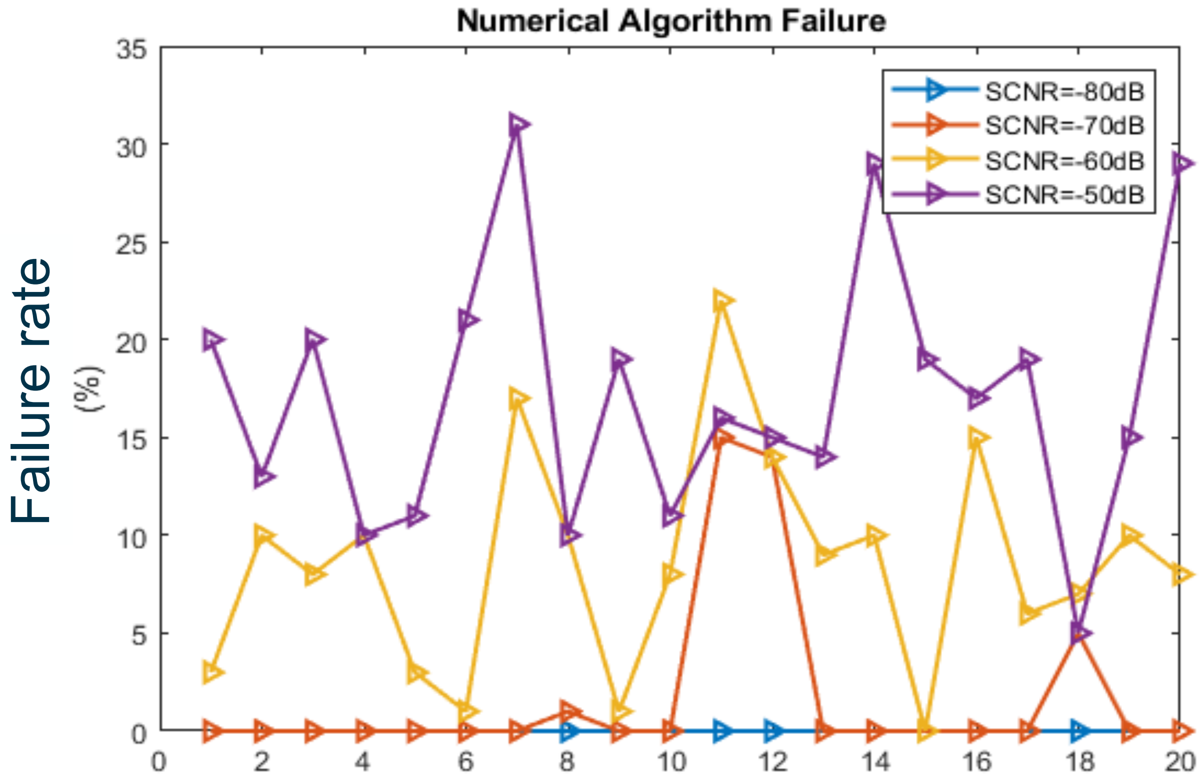




Average solution performs much better than Chen.
It is almost unaffected by noise.
Better than the numerical method except at highest SCNR.

Average solution performs better than Chen.
It is almost unaffected by noise.
Always better than the numerical method.

The numerical algorithm may not converge



Test for convergence over 20 different system error realisations as SCNR changes.

- 100% convergence success only for highest SCNR

Why does the Average solution do so well?

The system we are dealing with is

$$\boldsymbol{M_i = RFT_p S_i R_P FT + N_i}$$

with $\boldsymbol{i}$ = 1 – 4, so 16 complex measurements. For the unknowns we have

- 6 complex system errors
- 6 complex PARC errors
- 16 complex noise terms
- 1 real Faraday rotation angle.

This looks analytically intractable

BUT

a 1st order analysis of the Average scheme is possible, if all 2nd order terms are neglected (i.e. all products of errors, such as $\delta_1 \delta_3$).

1^{st} order analysis: $\hat{\Omega}$, $\hat{f}_i$ and $Im(\hat{\delta}_i)$ are unbiased

The estimate of Ω is **unbiased** with variance

$$V_{\hat{\Omega}_A} = \frac{1}{4}V_\delta(1 - Re[\rho]) + \frac{\langle S^2 \rangle}{16}V_{\varepsilon T} + \frac{1 + \langle C^2 \rangle}{8}V_{\delta T} + \frac{V_N}{4}$$

The estimates of f_i are both **unbiased** with variance

$$V(\hat{f}) = \frac{1+\langle C^2 \rangle}{2}V_{\varepsilon T} + \langle S^2 \rangle V_{\delta T} + V_N$$

Their real and imaginary parts both have variance equal to half this value.

The estimates of the imaginary parts of δ_i are all **unbiased** with variance

$$\mathrm{V}\left(\mathrm{Im}(\hat{\delta}_i)\right) = \frac{\langle S^2 \rangle}{8}\mathrm{V}_{\varepsilon T} + \frac{1+\langle C^2 \rangle}{4}\mathrm{V}_{\delta T} + \frac{\mathrm{V}_N}{2}$$

- $C = \cos 2\Omega$ and $S = \sin 2\Omega$
- δ_2 and δ_4 have complex correlation coefficient ρ
- T indicates "transponder"

1^{st} order analysis: $Re(\widehat{\delta_i})$ is biased, i = 2,4, unbiased i =1,3

The estimates of the real part of δ_1 and δ_3 are **unbiased** with variance

$$V\left(Re\left(\hat{\delta}_{\mathrm{i}}\right)\right) = \frac{1}{4}(1 - Re[\rho])V_\delta + \frac{5}{16}\langle S^2\rangle V_{\varepsilon T} + \frac{1}{8}\langle 1 + 5C^2\rangle V_{\delta T} + \frac{3}{4}V_N$$

The estimates of the real parts of δ_2 and δ_4 are **biased** with mean values

$$\langle Re(\hat{\delta}_2)\rangle \sim = \frac{1}{2}Re(\delta_2[1 + \rho^*])$$

$$\langle Re(\hat{\delta}_4)\rangle \sim = \frac{1}{2}Re(\delta_4[1 + \rho])$$

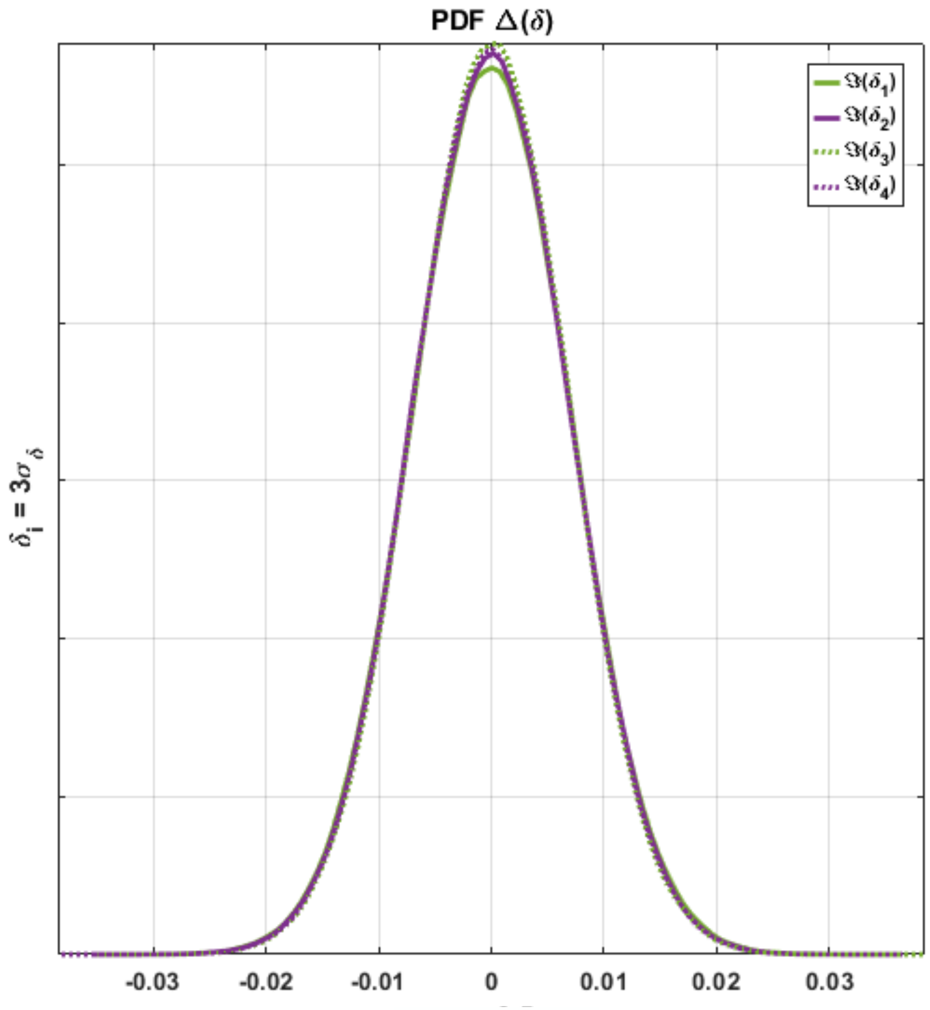
The bias gets smaller as $\rho \rightarrow 1$ (if real).

Both have the same variance

$$V\left(Re\left(\hat{\delta}_{\mathrm{i}}\right)\right) = \frac{1-|\rho|^2}{8}V_\delta + \frac{\langle S^2\rangle}{16}V_{\varepsilon T} + \frac{\langle 1+C^2\rangle}{8}V_{\delta T} + \frac{1}{4}V_N$$

For system error variances typical of those expected for BIOMASS, the noise variance is much smaller than the other terms.
The average solution is almost unaffected by clutter + noise.

Histograms of estimates of $\mathrm{Im}(\delta_i)$ using full simulation

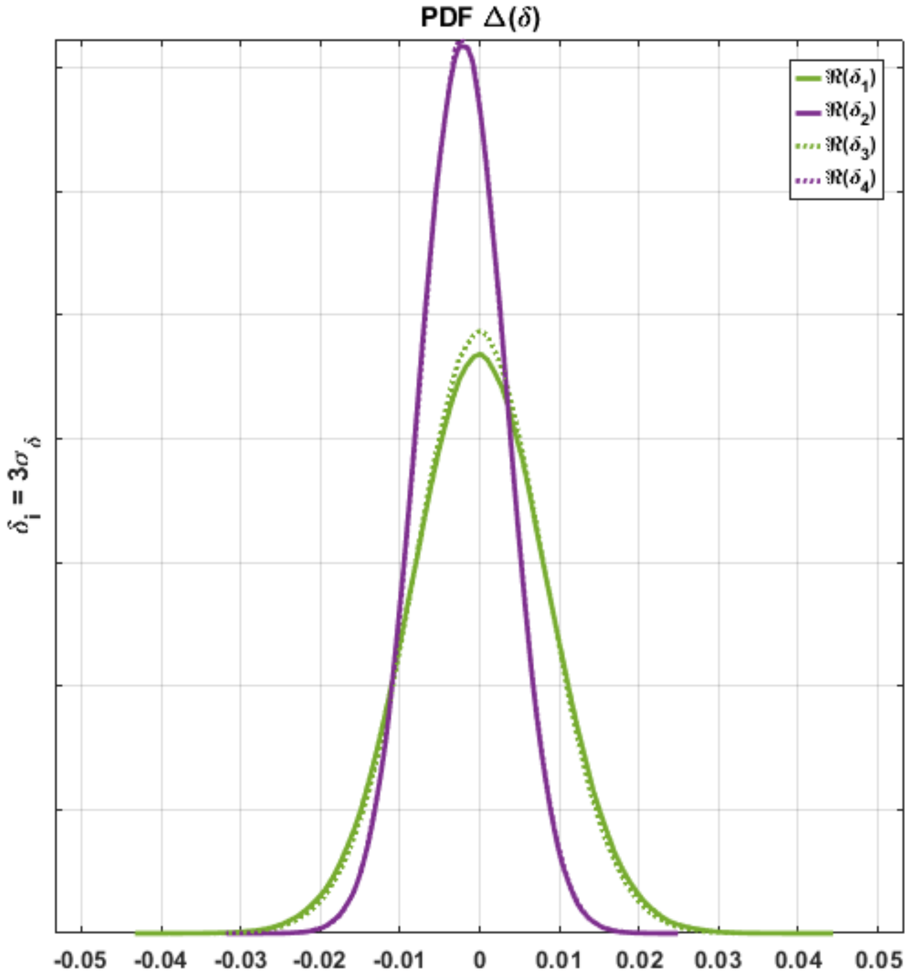


ρ = 0.5

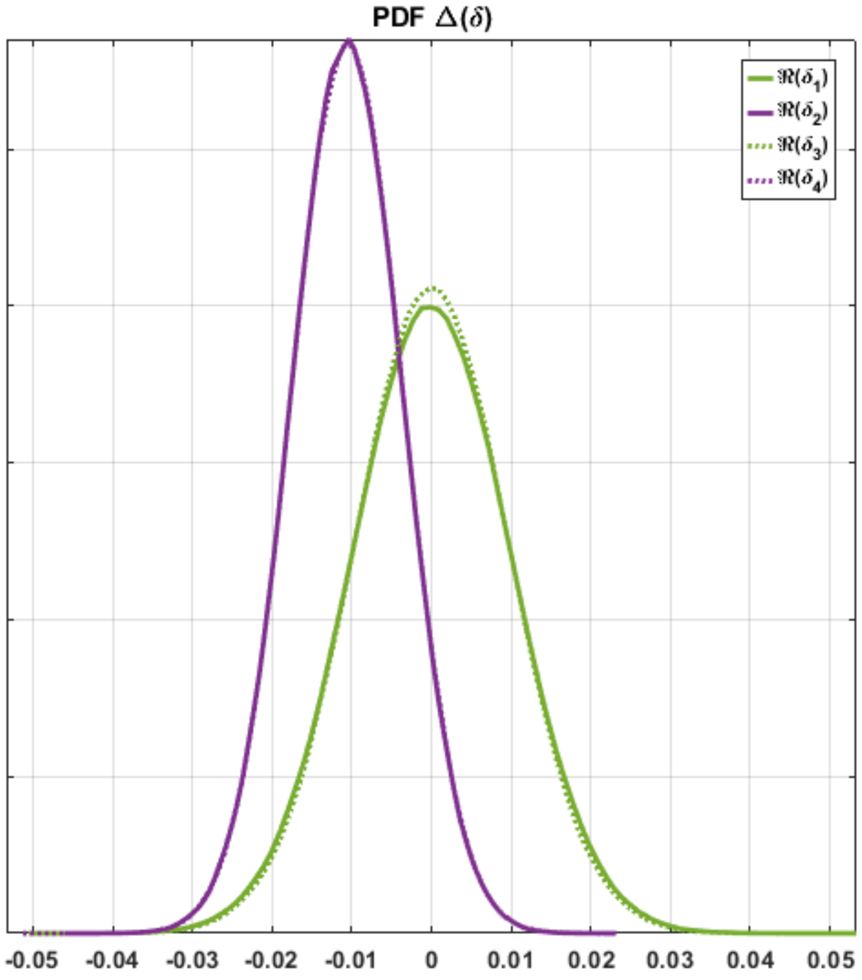
$\hat{\boldsymbol{\delta}}_i$ is unbiased for all $\boldsymbol{i}$.

Var(Im($\hat{\boldsymbol{\delta}}_i$)) is equal for all $\boldsymbol{i}$.

Histograms of estimates of $Re(\hat{\delta}_i)$ using full simulation



ρ = 0.9



ρ = 0.5

$\hat{\delta}_i$ is biased $i = 2, 4$; unbiased $i = 1, 3$.

The bias decreases as $\rho \to 1$.
If ρ is known, the bias can be corrected.

$\text{Var}(\text{Re}(\hat{\delta}_i))$ is equal $i = 1, 3$ & $i = 2, 4$.

Key conclusions

1. The performance of analytic methods to solve the PARC system for system errors and Faraday rotation is comparable to or better than numerical optimisation.

2. Numerical methods may not converge.

3. First order analysis indicates biases in the estimates of 2 of the cross-talk terms.

4. Correcting this bias should reduce the errors in the cross-talk terms: this analysis still needs to be performed.

Simulation parameters

Values used for the plots:

$V_\delta = -31$ dB

$V_\varepsilon = -25$ dB

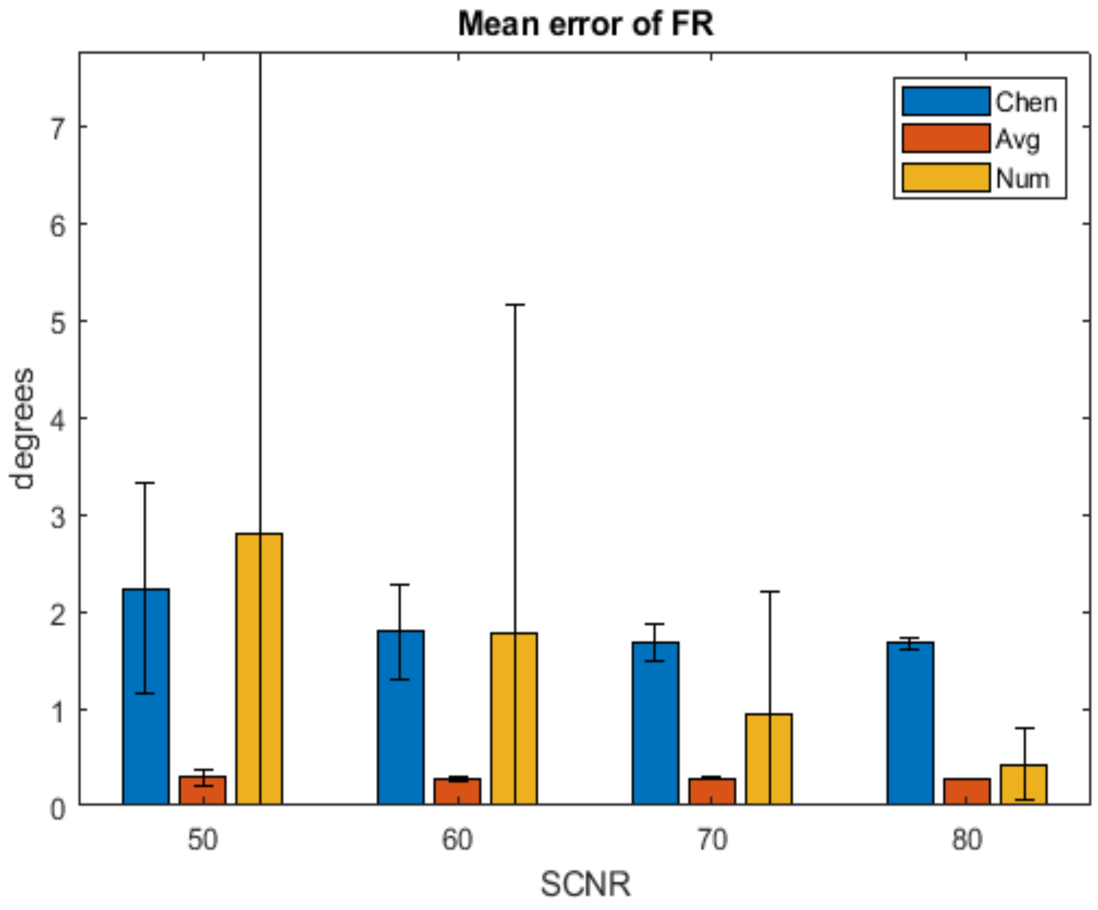
$V_{\delta T} = -35$ dB

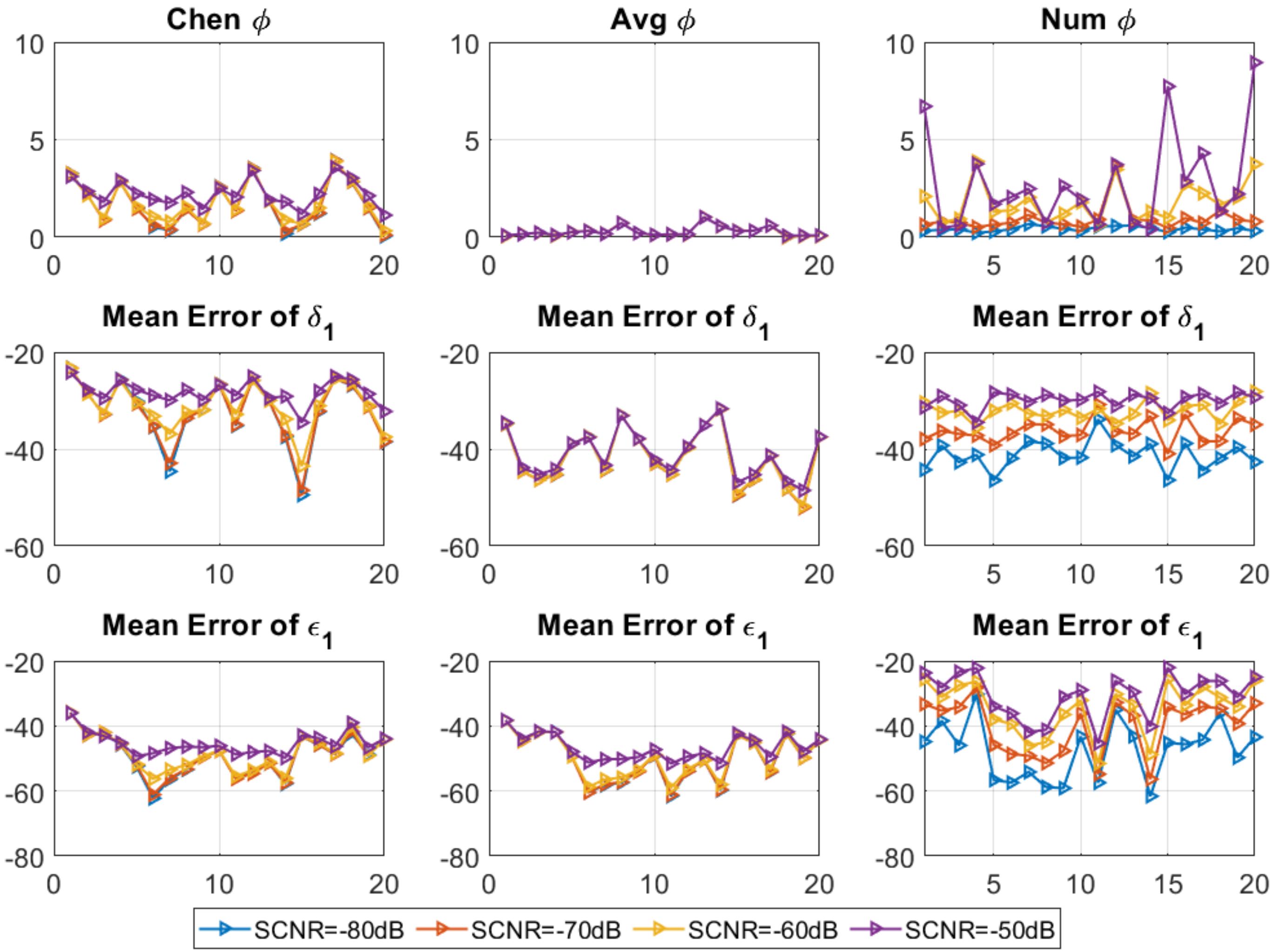
$V_{\varepsilon T} = -35$ dB

$V_N = -50$ dB

$\rho = 0.9$ and 0.5

Ω is normally distributed with mean 30º and standard deviation 10º.


Mean error of FR
Chen
Avg
Num
degrees
0
1
2
3
4
5
6
7
50
60
70
80
SCNR


Chen ϕ
Avg ϕ
Num ϕ
Mean Error of δ_1
Mean Error of δ_1
Mean Error of δ_1
Mean Error of ϵ_1
Mean Error of ϵ_1
Mean Error of ϵ_1
SCNR=-80dB
SCNR=-70dB
SCNR=-60dB
SCNR=-50dB

Assumptions in following results

For simplicity, we assume that

- the system cross-talk errors (i.e. the δ_i) all have variance V_δ
- the system channel imbalance errors (i.e. the ε_i) all have variance V_ε
- δ_2 and δ_4 have complex correlation coefficient ρ
- the PARC cross-talk errors (i.e. the δ_{Ti}) all have variance $V_{\delta T}$
- the PARC channel imbalance errors (i.e. the ε_{Ti}) all have variance $V_{\varepsilon T}$
- the noise terms all have variance V_N
- Ω is a random variable; $C = \cos 2\Omega$; $S = \sin 2\Omega$